Integer Linear Algebra Problem

Let $\mathbb{Z}$ be the integers, and for $\ell, u \in \mathbb{Z}$ with $\ell < u$ let $[\ell, u)$ denote the set $\{\ell, \ell+1, \dots, u-1\}$.

Assume you have a subset of $\mathbb{Z}^n$, i.e., multi-indices $(x_0, \dots, x_{n-1})$, given by the cross-product

$$I_n = [\ell_0, u_0) \times [\ell_1, u_1) \times \cdots \times [\ell_{n-1}, u_{n-1}) \tag{1}$$

that does not contain the zero vector, and an $m \times n$ matrix A with integer coefficients.

The question is how to determine whether there is an $x \in I_n$ with $Ax = 0$.

This is equivalent to determining whether there is an $x \in I_n$ with $\|Ax\|^2 = 0$, which is a quadratic mixed integer optimization problem with constraints (and no continuous variables).

Appendix

In this appendix we start with the real problem I want to solve and manipulate it into something that's maybe a bit more standard.

We'll be working with multi-indices, finite vectors of integers, e.g., $x = (x_0, \dots, x_{n-1})$, $\ell = (\ell_0, \dots, \ell_{n-1})$, $u = (u_0, \dots, u_{n-1})$, ...

We write $\ell < u$ to mean $\ell_i < u_i$ for $i = 0, \dots, n-1$, and define for $\ell < u$ the multi-dimensional half-open interval of multi-indices $[\ell, u) = [\ell_0, u_0) \times \cdots \times [\ell_{n-1}, u_{n-1})$.

Given an $m \times n$ matrix A of integers, an m-vector of integers b, and an interval $[\ell, u)$ with $\ell < u$, we want to determine whether the mapping $x \to Ax + b$ is one-to-one on $[\ell, u)$, i.e., whether

$$(\forall x, y \in [\ell, u) \mid x \neq y) \quad Ax + b \neq Ay + b.$$

E.g., if $A = (\,10 \quad 1\,)$, $b = (0)$, $\ell = (-4, -4)$, and $u = (100, 4)$, then A is one-to-one on $[\ell, u)$, but definitely not one-to-one on all of $\mathbb{Z}^2$.

We can rewrite this condition as

$$(\forall x, y \in [\ell, u) \mid x \neq y) \quad Ax \neq Ay.$$

For $x \in [\ell, u)$, we write $x = \ell + \hat{x}$, where $\hat{x}$ is now in $[0, \hat{u})$, $\hat{u} = u - \ell$, so we want to know whether

$$(\forall \hat{x}, \hat{y} \in [0, \hat{u}) \mid \hat{x} \neq \hat{y}) \quad A(\hat{x} + \ell) \neq A(\hat{y} + \ell),$$

or

$$(\forall \hat{x}, \hat{y} \in [0, \hat{u}) \mid \hat{x} \neq \hat{y}) \quad A\hat{x} + A\ell \neq A\hat{y} + A\ell,$$

or

$$(\forall \hat{x}, \hat{y} \in [0, \hat{u}) \mid \hat{x} \neq \hat{y}) \quad A\hat{x} \neq A\hat{y},$$

or

$$(\forall \hat{x}, \hat{y} \in [0, \hat{u}) \mid \hat{x} \neq \hat{y}) \quad A(\hat{x} - \hat{y}) \neq 0.$$

I claim that $\{\hat{x} - \hat{y} \mid \hat{x}, \hat{y} \in [0, \hat{u})\}$ is the (open) interval $(-\hat{u}, \hat{u})$. The mapping is simply to put the nonnegative components of elements in $(-\hat{u}, \hat{u})$ into $\hat{x}$ and the negative components into $-\hat{y}$.

So we're going to drop the hats from variables, and we want to know whether

$$(\forall x \in (-u, u) \mid x \neq 0) \quad Ax \neq 0.$$

So that's the problem. I decided to write $(-u, u) \setminus \{0\}$ as

$$\bigcup_{i=0,\dots,n-1} \left((-u, (u_0, \dots, u_{i-1}, -1, u_{i+1}, \dots, u_{n-1})) \cup ((-u_0, \dots, -u_{i-1}, 1, -u_{i+1}, \dots, -u_{n-1}), u) \right),$$

i.e., as a union of intervals not containing the origin.

Hence my question: Given an interval of multi-indices that doesn't contain the origin and a matrix of integers A, does A have a zero for arguments in the given interval.

Brad